HARVESTING WITH SMOOTHING COSTS

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I. INTRODUCTION

Management plans for renewable resources will not be effective unless they reflect both the economic realities faced by the fishermen and also the vagaries of stock growth. Recent research on the effect of stochastic environmental factors on stock growth, such as currents, thermal isoclines or wind (talks given at the Tuna Conference, Lake Arrowhead, CA, September 1976) suggests that deterministic models will not suffice to manage these resources. When stock growth varies in a probabilistic fashion, deterministic concepts such as "maximum sustained yield" lose meaning.

Fortunately, the mathematics exists to determine optimal policies for stochastic harvesting models (see Jacquette 1972, 1974; Reed 1974, 1975; Mendelssohn and Sobel 1977). The most complete treatment is in Mendelssohn and Sobel (1977), and the purpose of this paper is to extend their results to include costs due to fluctuations in harvest size, called smoothing costs.

For stochastic versions of many of the standard production models, with linear (discounted) costs, an optimal policy in each period t is described by a "base stock size" x_t^o. If the stock size at the beginning of the period is greater than x_t^o, then it is optimal to harvest to x_t^o. Otherwise it is optimal not to harvest. This policy is summarized in Table 1.

Stock size at beginning of period	Optimal harvest size	Stock size at end of period
$x > x_t^o$	$x - x_t^o$	x_t^o
$x \leq x_t^o$	0	x

As the number of periods in the planning horizon tends to infinity, the sequence of x_t^o's tends to a single base stock size x^o, which describes an optimal policy in each period in the infinite horizon model.

It is curious that a similar idea of managing for stock size rather than catch size when growth is stochastic has arisen from the practical experience of several people involved in fisheries research (see Gulland and Boerema 1973 or Radovich 1975). Managing for stock size probably implies that an independent estimate of stock size is available, (which is true for many coastal and demersal fisheries) and will probably delineate where the results we present will be useful.

Let us follow what might happen when following a base stock policy over an infinite planning horizon. Stock size each year is being kept as close to x^o as possible. However, catch size is fluctuating, the amount of fluctuation depending on the variances of the random elements and also on how much the random elements affect the growth of the stock. Downward fluctuations in the harvest may cause economic hardship in the fishery due to insufficient allowable catch to insure everyone a reasonable "profit." Upward fluctuations in the catch may cause the fishermen to "gear up," which will find them in debt in later periods when the catch quota is lowered. The problem is to manage around the base stock size in a manner that reflects the costs due to fluctuations in catch, which we call smoothing costs.

Suppose there is a cost (real or subjective) to an increase in the allowable catch, given by γ for each unit of increase. Similarly, suppose there is a cost ε for each unit of decrease in the catch size between periods. Let z_t be the catch in period t and z_{t-1} the catch in period t-1. Then the net benefit of z_t is:

$$\begin{cases} p \cdot z_t - \varepsilon \cdot (z_{t-1} - z_t) & z_t < z_{t+1} \\ p \cdot z_t & z_t = z_{t-1} \\ p \cdot z_t - \gamma \cdot (z_t - z_{t-1}) & z_t \geq z_{t-1} \end{cases} \tag{1.1}$$

In the literature on inventory and production management, cost structures such as equation (1.1) are analyzed under titles like "Production models with smoothing costs." The similarities between harvesting models and inventory models has been noted elsewhere (Mendelssohn and Sobel 1977), and the results presented here can be seen to be "mirror images" of results in Beckmann (1961) and Sobel (1969, 1971).

Section II formally describes the model to be analyzed. Section III describes the results for a finite planning horizon. Section IV extends these results to an infinite planning horizon. Section V discusses possible extensions of the results. Only the theorems are presented in the main body of the paper. The proofs can be found in the appendix.

For a real-valued function f that maps a subset of $\mathbb{R}$ into $\mathbb{R}$, f^1 denotes the derivative of the function. If a real-valued function h maps a subset of $\mathbb{R}^2$ into $\mathbb{R}$, $h^{[1]}$ denotes the partial derivative with respect to the first argument, and $h^{[2]}$ the partial derivative with respect to the second argument.

II. THE MODEL

The resource is being managed for a planning horizon of T periods, $0 < T \leq \infty$. The periods are subscripted by $t = 1, 2, \ldots, T$ or alternatively by $n = T - t + 1$, the number of periods remaining in the planning horizon. At the beginning of each period t, an initial stock size x_t is observed, a harvest of size z_t is taken during period t, and a stock size y_t is left at the end of the period after harvesting has ceased. The stock size at the beginning of period t + 1, x_{t+1}, is a random function of the stock size at the end of period t and an exogenous random variable; i.e.

$$x_{t+1} = s[y_t, D_t]$$

where $s[y_t, D_t]$ is the "transition function" and $D_1, D_2, \ldots, D_T$ are independent, identically distributed random variables, distributed as the generic random variable D. The proofs do not depend on D being a scalar random variable; thus $D_1, D_2, \ldots, D_T$ can be a sequence of independent, identically distributed random vectors, distributed as the generic random vector D. This implies that the transition function is capable of including any number of environment variables.

For each fixed realization of the random variable D_t (denoted by d), $s[\cdot, d]$ is assumed to be concave and continuous. Many of the standard production models satisfy these assumptions. A harvest of $z_t = x - y_t$ produces a return of $G(x - y)$. If z_t is greater than z_{t-1}, there is a cost of $\gamma \cdot (z_t - z_{t-1})$, $\gamma \geq 0$. If z_t is less than z_{t-1}, there is a cost $\varepsilon \cdot (z_{t-1} - z_t)$, $\varepsilon \geq 0$. The total one period return in period t is:

$$\begin{cases} G(z_t) - \gamma \cdot (z_t - z_{t-1}) & z_t > z_{t-1} \\ G(z_t) - \varepsilon \cdot (z_{t-1} - z_t) & z_t < z_{t-1} \\ G(z_t) & z_t = z_{t-1} \end{cases}$$

and this return is discounted by a factor α^{t-1}, $0 < \alpha \leq 1$.

The model as described is a two state model. At the beginning of each period, an initial population size x is observed, and a harvest size z from the previous period also is known. If (x, z) is the observed state in period t, then Beckmann (1961) shows that equation (2.1) can be rewritten as:

$$G(z_t) - e \cdot (z - z_t) - c \cdot |z_t - z| \tag{2.2a}$$

$$= G(x - y_t) - e \cdot \Big(z - (x - y_t)\Big) - c \cdot |x - y_t - z| \tag{2.2b}$$

where $e = \frac{(\gamma + \varepsilon)}{2}$ and $c = \frac{(\gamma - \varepsilon)}{2}$. The dynamic optimization problem is to choose a feasible sequence $\{y_t\}$ that:

$$\text{maximizes } E\left\{\sum_{t=1}^{T}\left[\alpha^{t-1}g(x, y, z)\right]\right\}$$

$$\text{subject to } 0 \leq y_t \leq x_t;\ 0 \leq z_t \leq x_t;\ x_{t+1} = s[y_t, D_t] \tag{2.3}$$

$$x_t, y_t, z_t, \geq 0$$

and $g(\cdot, \cdot, \cdot)$ is given by equation (2.2b). It follows from Sobel (1969) that if an optimal policy exist, it must be the solution to the following system of recursive equations:

(where n = the number of periods remaining, is the subscript).

$$f_o(\cdot,\cdot) \equiv 0 \tag{2.4}$$

$$f_n(x, z) = \sup_{0 \leq y \leq x}\left\{J_n(x, y) - c \cdot |x - y - z| - e \cdot z\right\}$$

where $J_n(x, y) = G(x - y) + \alpha E\left\{f_{n-1}(s[y, d], x - y)\right\}$

The problem described by the dynamic program (2.4) differs from the typical bioeconomic model (see Clark 1976) in three important ways. First, a state (x, z) in any future period is never reached with certainty, but only with a given probability, unless $x = 0$ or $y = 0$. Therefore the phase-plane diagrams used to describe optimal policies (as in Clark 1976) have no meaning in this context. Instead, it is necessary to define, for each period n, a function $A_n(x, z)$, called the "policy function." The function $A_n(x, z)$ describes a decision y for each (x, z) such that if (x, z) were the observed state in period n, then choosing $y_n = A_n(x, z)$ and following the policy described by $A_{n-1}(\cdot)$, $A_{n-2}(\cdot)$, ..., $A_1(\cdot)$ maximizes the expected total return from period n to the end of the planning horizon.

Second, the decision variable is stock size, not fishing effort. Stock size is a function of the resource, effort a function of the industry that utilizes the resource. Moreover, when effort is the decision variable, the decision each period has no effect on the range of feasible decisions in the following periods. When population size is the decision, the stock size the next period, a random variable, constrains the decision in that period. Therefore the decision this period carries over into the future.

Third, equilibrium points, or maximum sustained yield points, and their properties, which are the main interest in deterministic analysis, are of little interest in a stochastic model. Rather, the interest becomes to describe properties of the policy $A_n(x, z)$, that lend insight into how a harvest varies in time (i.e., as n varies) and in state (i.e., as x or z vary), and hopefully in order to increase computational efficiency. This is the purpose of the next three sections.

III. THE FINITE HORIZON MODEL

In this section, it is assumed that the planning horizon extends over a finite number of periods. The single-period return function, given by equation (2.2b) and the transition function $s[\cdot, d]$ are both concave, and have at least one-sided derivatives. Analysis of the model would be facilitated if these properties are also true for $J_n(x, y)$ and $f_n(x, z)$ in each period.

<u>Lemma 1</u> For each n:

(i) $J_n(x, y)$ is concave and continuous on the set $C = \{(x, y): x \in X;\ 0 \leq y \leq x\}$, and $J_n(x, y) - c \cdot |x - y - z|$ is jointly concave in (x, y, z).

(ii) $f_n(x, z)$ is concave and continuous, $f_n(\cdot, z)$ is nondecreasing and $f_n(x, \cdot)$ is nonincreasing.

$\square$

To describe an optimal policy, we need to make assumptions about $G(z)$, the immediate return from a harvest of size z. The following assumptions are discussed in Mendelssohn (1976) or Mendelssohn and Sobel (1977):

$$\begin{aligned}
&\text{(i) } G(x, y) \text{ is concave and continuous on the set} \\
&\qquad C = \{(x, y): x \in X;\ 0 \le y \le x\} \\
&\text{(ii) } G(\cdot, y) \text{ is nondecreasing} \\
&\text{(iii) } G^{[2]}(\cdot, y) \text{ is nondecreasing} \\
&\text{(iv) } G \text{ is nonnegative (uniformly bounded below) on } C.
\end{aligned} \tag{3.1}$$

Further, consider the following two functions:

$$\begin{aligned}
y_n^1(x) &= \sup \left\{y:\ J_n^{[2]}(x, y) \ge -c;\ 0 \le y \le x\right\} \\
\bar{y}_n^2(x) &= \sup \left\{y:\ J_n^{[2]}(x, y) \ge c;\ 0 \le y \le x\right\}
\end{aligned} \tag{3.2}$$

Let us consider the meanings of these different definitions. $y_n^1(x)$ is a feasible y for each x, such that if the harvest size is increased, it is optimal to increase it to $x - y_n^1(x)$. Similarly, $y_n^2(x)$ is such that if the harvest size decreases, it is optimal to decrease to $x - y_n^2(x)$. It is shown in Mendelssohn (1976), or Mendelssohn and Sobel (1977) that an optimal policy function $A_n(x)$ for the problem with no smoothing costs has the following properties:

$$0 \le \frac{d}{dx} A_n(x) \le 1$$

This leads us to suspect that it might be true that, $0 \le \frac{d}{dx} y_n^i(x) \le 1$ for i = 1, 2. Theorem 1 is the main theorem of this paper. It states that this conjecture is correct, and shows how this can be used to describe an optimal policy function.

Theorem 1 (a) Let $G(x, y)$, the one period benefit from harvesting, be given by equation (3.1). Then there exists two functions $y_n^1(x)$ and $y_n^2(x)$, defined in equations (3.2) such that the value of an optimal harvesting strategy is given by:

$$f_n(x, z) = \begin{cases} J_n(x, x) - c \cdot z & : x < y_n^2(x) \\ J_n(x, y_n^2(x)) + c \cdot (x - y_n^2(x) - z) & : x - z < y_n^2(x) \\ J_n(x, x - z) & : y_n^2(x) \leq x - z \leq y_n^1(x) \\ J_n(x, y_n^1(x)) - c \cdot (x - y_n^1(x) - z) & : x - z > y_n^1(x) \end{cases} \qquad \begin{matrix} \text{Region} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{matrix}$$

an optimal harvesting strategy is given by:

$$A_n(x; z) = \begin{cases} x & : x < y_n^2(x) \\ y_n^2(x) & : x - z < y_n^2(x);\ x \geq y_n^2(x) \\ x - z & : y_n^2(x) \leq x - z \leq y_n^1(x) \\ y_n^1(x) & : x - z > y_n^1(x) \end{cases} \qquad \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{matrix}$$

(b) Let $A_n(x)$ be the policy function when $\gamma, \varepsilon \equiv 0$. Then, under the assumptions of part (a), the following inequalities are true:

(i) $x - y_n^1(x) \leq x - A_n(x) \leq x - y_n^2(x)$

(ii) $0 \leq A_n^{[1]}(x, z) \leq 1$

(iii) $-1 \leq A_n^{[2]}(x, z) \leq 0$

(iv) $0 \leq y_n^{2\prime}(x) \leq 1$

(v) $0 \leq y_n^{1\prime}(x) \leq 1$

□

Theorem 1 provides a welter of results at once. Figure 1 summarizes visually an optimal policy. The abscissa is the value of x, the ordinate is the value of z. What is read off the graph is the value of $z_t^* = x - A_n(x, z)$, by finding the corresponding value on the z-axis. The left-most dotted line shows the limits of the constraint $x_t \geq z_t$. The middle dotted line is the policy $z = x - A_n(x)$.

A second way of summarizing an optimal policy is given in Table 2. The first column considers the value of the remaining stock size if the old harvest were followed. The second column gives the optimal harvest size, and the third column gives the remaining stock size.

Stock size if harvest unchanged	Optimal harvest	Size of remaining stock
$x - z < y_n^2(x)$; $x < y_n^2(x)$	0	x
$x - z < y_n^2(x)$; $x \geq y_n^2(x)$	$x - y_n^2(x)$	$y_n^2(x)$
$y_n^2(x) \leq x - z \leq y_n^1(x)$	z	$x - z$
$y_n^1(x) < x - z$	$x - y_n^1(x)$	$y_n^1(x)$

Consider each region. In region I, $J_n^{[2]}(x, x) > - c$. In other words, it would be desirable to decrease the harvest beyond zero. Since this isn't feasible, $A_n(x, z) - x$ is an optimal policy. In region II, either a decrease in harvest size is absolutely necessary, due to the constraint $x \geq z_t$, in which case

$A_n(x, z) = y_n^2(x)$ is optimal, or else at $y_n^2(x)$ the marginal gain from decreasing the harvest size just equals the marginal cost. In region III, the marginal cost of any change in harvest size is greater than the increase in value. In region IV, $y_n^1(x)$ is the point where the marginal increase in value due to an increase in harvest size just equals the marginal change in the cost.

Clearly the choice of c (respectively of γ, ε) influences an optimal policy to a large degree, as the width of region III will depend on these values. In many fisheries contexts, c represents a subjective weighting against undesirable events rather than a true monetary cost. Therefore, care should be used when actually choosing a value for c. For numerical problems, several values of c might be tried, in order to see how sensitive an optimal policy is to changes in c. We conjecture, but have not proven, that the change in an optimal policy is greater when changes are made in small values of c than when changes are made in large values of c. The intuition behind the conjecture is that as smoothing costs first enter the problem, an optimal policy "jumps" to include these costs. But as c gets larger and larger, an optimal policy becomes heavily weighted towards not changing the allowable catch, and therefore changes in the value of c should have little effect on an optimal policy.

IV. INFINITE HORIZON MODEL

In this section, n, the number of periods remaining in the planning horizon, approaches infinity. As in the base stock size policy, it would be convenient if as n approaches infinity, that an optimal policy approach a stationary policy, that is one that doesn't depend on the period. If such limits exist, consider the following five functions:

$$
\begin{aligned}
A(x, z) &= \lim_{n\to\infty} A_n(x, z) \\
y^1(x) &= \lim_{n\to\infty} y_n^1(x) \\
y^2(x) &= \lim_{n\to\infty} y_n^2(x) \\
f(x, z) &= \lim_{n\to\infty} f_n(x, z) \\
J(x, y) &= \lim_{n\to\infty} J_n(x, y)
\end{aligned}
\tag{4.1}
$$

Theorem 2 states conditions for which these limits exist, and for which the functions described are the functions for the infinite horizon problem.

<u>Theorem 2.</u> (a) Assume s[•, d] is nondecreasing for each fixed value of d. Then:

$$
\begin{aligned}
y_{n+1}^1(x) &\geq y_n^1(x) \\
y_{n+1}^2(x) &\geq y_n^2(x) \\
A_{n+1}(x, z) &\geq A_n(x, z) \\
f_{n+1}(x, z) &\geq f_n(x, z) \\
J_{n+1}^{[2]}(x, y) &\geq J_n^{[2]}(x, y)
\end{aligned}
$$

(b) Assume also that c, e are finite and that G(z) is bounded above by a finite number M such that $\lim_{t\to\infty} \alpha^{t-1} M \to 0$. Then there exists functions $A(x, z)$, $y^1(x)$, $y^2(x)$, $f(x, z)$, and $J(x, y)$ defined in equation (4.1) such that an optimal policy for the infinite horizon problem is given by:

$$A(x, z) = \begin{cases} x: \; x < y^2(x) & \text{I} \\ y^2(x): \; x - z \le y^2(x); \; x \ge y^2(x) & \text{II} \\ x - z: \; y^2(x) \le x - z \le y^1(x) & \text{III} \\ y^1(x): \; y^1(x) \le x - z & \text{IV} \end{cases} \qquad \text{Region}$$

the value of an optimal policy is:

$$f(x, z) = \begin{cases} J(x, x) + c \cdot z & \text{I} \\ J\left(x, y^2(x)\right) + c \cdot \left(x - y^2(x) - z\right) & \text{II} \\ J(x, z) & \text{III} \\ J\left(x, y^1(x)\right) - c \cdot \left(x - y^1(x) - z\right) & \text{IV} \end{cases} \qquad \text{Region}$$

and the following inequalities are valid:

(i) $x - y^1(x) \le x - A(x) \le x - y^2(x)$

(ii) $0 \le A^{[1]}(x, z) \le 1$

(iii) $-1 \le A^{[2]}(x, z) \le 0$

(iv) $0 \le y^{2\prime}(x) \le 1$

(v) $0 \le y^{1\prime}(x) < 1$

$\square$

The interpretation of theorem 2 is essentially that of theorem 1, so it will not be repeated. Over an infinite horizon, the problem reduces to one of calculating two functions, $y^1(x)$ and $y^2(x)$, both of which have derivatives bounded between zero and one. Sobel (1971) discusses how to take advantage of similar structure in an optimal policy function in order to efficiently calculate an optimal policy.

V. DISCUSSION AND SUMMARY

We have seen that when smoothing costs are added to a harvesting problem, a three region optimal policy arises. These regions smooth out the fluctuations in the catch while still being concerned about effects to the stock. The exact balance that is achieved will depend on the relative values of the smoothing costs to the "market" value of the harvest.

Certain immediate extensions suggest themselves. $G(z)$ and $s[y, d]$ often are period dependent functions, reflecting perhaps seasonal fluctuations in price or growth. Theorem 1 is valid as stated as long as the assumptions are valid for each G_n and s_n. However, theorem 2 is not necessarily valid as stated in the nonstationary case.

Also, for most fisheries, a more realistic description of smoothing costs assumes that for a certain level of change, there is no smoothing cost, and that the per unit cost decreases as the change begins to get large. That is, some fluctuations are too small to notice, and some are so large that the "damage" has already been done, and the increased cost due to an even larger change is minimal.

In practice, some if not all of the functions will be difficult to estimate. However, experience with and knowledge of a fishery may make it possible for a manager to put approximate bounds on the three regions. Certainly knowing the form of a policy is a step towards obtaining methods to put numbers in that form. It is hoped that this initial effort will stimulate such applied research.

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APPENDIX

It is convenient to prove lemma 1 and theorem 1 together, and then to prove theorem 2 as an extension of the results of theorem 1. That will be our approach.

Proof of Lemma 1 and Theorem 3

At $n = 1$, the return is $G(x-y) - e \cdot (z-(x-y)) - c \cdot |x-y-z|$ which by the assumptions on G is jointly concave in (x, y, z). From a theorem in Iglehart (1965), which he attributes to A. F. Veinott, Jr. and is a generalization of a theorem due to Dantzig (1955), it follows that $f_1(x, z)$ is concave and (left)-continuous in (x, z).

As an induction hypothesis, assume $f_1, f_2, \ldots, f_{n-1}$ are concave and continuous, and that $f_{n-1}(\cdot, z)$ is nondecreasing and $f_{n-1}(x, \cdot)$ is nonincreasing (which we will prove shortly). This implies that $\alpha Ef_{n-1}(s[y, d], x - y)$ is jointly concave in (x, y) and that $G(x - y) - e(z - (x - y) - c \cdot |x - y - z| + \alpha Ef_{n-1}(s[y, d], x - y)$ is jointly concave in (x, y, z). Again, from the theorem in Iglehart, it follows that $f_n(x, z)$ is jointly concave in (x, z).

If the induction hypothesis is true for some n, then an optimal y can be found by taking the partial derivative with respect to y, and making it as close to zero as possible, subject to $0 \leq y \leq x$. Suppose $x - z \geq y_n^1(x)$. Then:

$$J_n^{[2]}(x,\ y_n^1(x)) + c = 0$$

or else $y_n^1(x) \equiv x$. Thus, if the conditions defining region IV are valid, then $y_n^1(x)$ is optimal.

Suppose $x \leq y_n^2(x)$. Then, since $z \geq 0$, $x - z < y_n^2(x)$, so that the harvest must be decreased. Again,

$$J_n^{[2]}(x,\ y_n^2(x)) = 0$$

or else $y_n^2(x) \equiv x$, so that $y_n^2(x)$ is an optimal decision. If $x \geq y_n^2(x)$, and $x - z < y_n^2(x)$, by the same reasoning, $y_n^2(x)$ again is an optimal decision.

Suppose $y_n^2(x) \leq x - z \leq y_n^1(x)$. Then:

$$-c \leq J_n^{[2]}(x,\ x - z) \leq c$$

At $y = x - z$, the derivative is $J_n^{[2]}(x,\ x - z)$. An increase in y will decrease the value, i.e., $J_n^{[2]}(x,\ x - z + \delta) + c \leq J_n^{[2]}(x,\ x - z)$ and $J_n^{[2]}(x,\ x - z - \delta) - c \geq J_n^{[2]}(x,\ x - z)$ for $\delta > 0$, which implies that $x - z$ is an optimal decision. If the theorem is true in period j, then:

$$f_j^{[1]}(x,\ z) = \begin{cases} J_j^{[1]}\left(x,\ y_j^2(x)\right) + c & \text{Region I + II} \\ J_j^{[1]}(x,\ x - z) + \left[J_j^{[2]}(x,\ x - z)\right]^+ & \text{Region III} \\ J_j^{[1]}\left(x,\ y_j^1(x)\right) - c & \text{Region IV} \end{cases}$$

$$f_j^{[2]}(x, z) = \begin{cases} -(e + c) & \text{Region I + II} \\ -J_j^{[2]}(x, x - z) - e & \text{Region III} \\ c - e & \text{Region IV} \end{cases}$$

which proves that $f_n(\cdot, z)$ is nondecreasing and $f_n(x, \cdot)$ is non-increasing.

To see where we are, we have shown that $J_1(x, y) - c \cdot |x - y - z| + e$ is concave in (x, y, z), which implies an optimal policy given in part (a), which implies $f_1(x, z)$ is concave and continuous, and that $f_1(\cdot, z)$ is nondecreasing, $f_1(x, \cdot)$ nonincreasing. Assume this for $f_1, f_2, \ldots, f_{n-1}$. Then $\alpha E f_{n-1}(s[y, d], x - y)$ is jointly concave in (x, y), $J_n(x, y) - c \cdot |x - y - z| + e \cdot z$ is jointly concave in (x, y, z). Again, this implies an optimal policy given by part (a), which in turn implies $f_n(x, z)$ is concave and continuous, $f_n(\cdot, z)$ is nondecreasing and $f_n(x, \cdot)$ is nondecreasing, which are the desired results.

Much of part (b) has already been proven. By the definition of the four regions, in regions I, II, IV $A_n(x, z)$ does not change with z, in region III the change is exactly equal to the change in z. This proves (iii), $-1 \leq A_n^{[2]}(x, z) \leq 0$. Note that claim (ii) implies claim (iv) and (v). To see this, in region III it is already true that $0 \leq A_n^{[1]}(x, z) \leq 1$. If this is true in all other regions, since $A_n(x, z)$ will equal either $y_n^1(x)$ or $y_n^2(x)$ in these regions, (ii) must imply (iv) and (v).

At period 1:

$$f_1^{[1]}(x,\ z) = \begin{cases} 0 & \text{I} \\ 0 & \text{II} \\ 0 & \text{III} \\ 0 & \text{IV} \end{cases} \qquad \text{Region}$$

because $f_1^{[1]}(x, z) = G'\left(x - A_1(x, z)\right) + e \,(\pm)\, c + \left[-G'\left(x - A_1(x, z)\right) - e(\mp)c\,\right]^+ = 0$

and

$$f_1^{[2]}(x,\ z) = \begin{cases} -(e + c) & \text{I} \\ -(e + c) & \text{II} \\ -G'\left(x - A_1(x)\right) - e & \text{III} \\ c - e & \text{IV} \end{cases}$$

Assume for periods 1, 2, ..., n-1 $f_j^{[1]}$ is zero. To show this implies $f_n^{[1]}$ is zero, consider the region II case first.

Then:

$$\begin{aligned} f_n^{[1]}(x, z) = {} & G'\left(x - A_n(x, z)\right) + e + c + \Big[-G'\left(x - A_n(x, z)\right) \\ & - e - c + \alpha E\Big\{f_{n-1}^{[1]}\left(s\left[A_n(x, z), d\right]\right) s^{[1]}[A_n(x, z), d] \\ & - f_{n-1}^{[2]}\left(s[A_n(x, z), d], x - y\right)\Big\}\Big]^+ \\ & + f_{n-1}^{[2]}\left(s[A_n(x, z), d], x - y\right) \\ = {} & \alpha E\Big\{f_{n-1}^{[1]}\left(s[A_n(x, z), d],\ x - y\right) s^{[1]}[A_n(x, z),\ d]\Big\} \\ = {} & 0 \end{aligned}$$

by the induction hypothesis and similarly for the other regions. Likewise

$$f_n^{[2]}(x, z) = \begin{cases} -(e + c) & \text{I + II} \\ -J_n^{[2]}(x, x - z) - e & \text{III} \\ c - c & \text{IV} \end{cases}$$

Therefore:

$$J_n^{[2]}(x, y) = G'(x - y) - e - \alpha E\left\{f_{n-1}^{[2]}\left(s[y, d], x - y\right)\right\}$$

which is increasing in x. This implies:

$$A_n(x, z) \leq A_n(x', z) \qquad \text{for } x' \geq x.$$

Put the problem in terms of x and the decision $z = x - y$. Then:

$$H^{[2]}(x, z) = G'(z) + e + \alpha E\left\{f_{n-1}^{[2]}\left(s[x - z, d], z\right)\right\}$$

which again is nondecreasing in x. This implies that:

$$x - A_n(x, z) \leq x' - A_n(x', z) \qquad \text{for } x' \geq x.$$

Together the two inequalities imply

$$0 \leq A_n(x', z) - A_n(x, z) \leq x' - x \qquad \text{for } x' > x.$$

which implies $\quad 0 \leq A_n^{[1]}(x, z) \leq 1.$

The final result is that:

$$x - y_n^1(x) \leq x - A_n(x) \leq x - y_n^2(x)$$

or equivalently:

$$y_n^2(x) \leq A_n(x) \leq y_n^1(x).$$

At period n:

$$J_n^{[2]}\left(x, A_n(x)\right) = -G'\left(x - A_n(x)\right) - e + \alpha E\left\{f_{n-1}^{[2]}\left(s[A_n(x), d], x - A_n(x)\right)\right\}$$

Assume as an induction hypothesis that:

$$\alpha E\left\{f_{n-1}^{[2]}\left(s[A_n(x), d], x - A_n(x)\right) - f'_{n-1}\left(s[A_n(x), d]\right) s^{[1]}[A_n(x), d]\right\}$$

$$\leq |c| \tag{A.1}$$

where $f_{n-1}(x)$ is the value function without smoothing costs. The inequality implies the result. However, the inequality is trivially true in period one. From the definition of $f_n^{[2]}(x, z)$, the result in period n implies inequality (A.1) for $f_n(x, z)$ which completes the proof.

□

Proof of Theorem 2

We assume the set X of feasible population sizes has a finite least upper bound.

Part (a). For fixed z:

$$f_1^{[1]}(x, z) - f_o^{[1]}(x, z) = f_1^{[1]}(x, z) \geq 0$$

$$f_1^{[2]}(x, z) - f_o^{[2]}(x, z) = f_1^{[2]}(x, z) \leq 0$$

As an induction hypothesis, assume for some n, that:

$$f_n^{[1]}(x, z) \geq f_{n-1}^{[1]}(x, z), \; f_n^{[2]}(x, z) \leq f_{n-1}^{[2]}(x, z)$$

Then:

$$J_{n+1}^{[2]}(x, y) - J_n^{[2]}(x, y) =$$

$$\alpha E\{(f_n^{[1]}(s[y, d], x - y) - f_{n-1}^{[1]}(s[y, d], x - y))s^{[1]}[y, d]$$

$$- (f_n^{[2]}(s[y, d], x - y) - f_{n-1}^{[2]}(s[y, d], x - y)) \geq 0$$

This implies that:

$$A_n(x, z) \geq A_{n-1}(x, z) \tag{A.6}$$

From the definition of $A_n(x, z)$, $A_{n-1}(x, z)$ given in theorem 1, $A_n(x, z) \geq A_{n-1}(x, z)$ must imply:

$$f^{[1]}_{n+1}(x, z) \geq f^{[1]}_n(x, z)$$

$$f^{[2]}_{n+1}(x, z) \leq f^{[2]}_n(x, z)$$

which completes the induction.

Part (b). We make the further assumption that $0 < \alpha < 1$. We have proven in part (a) that:

$$A_{n+1}(x, z) \geq A_n(x, z)$$

However, since the set X is bounded below by zero and has a finite upper bound by assumption, this implies that $\{A_n(x, z)\}$ approaches a limit $A(x, z)$. Moreover, since the one-period return is finite and bounded,

$$\alpha^{t-1} g(x, y, z) \to 0$$

as $t \to$ infinity, where $g(x, y, z)$ is given by:

$$G(x - y) - c \cdot |x - y - z| + e \cdot z.$$

It follows from Denardo (1967) that $\{f_n\}$ must converge to a limit $f(x, z)$ as n approaches infinity. Since both sequences $\{f_n(x, z)\}$ and $\{A_n(x, z)\}$ converge, of necessity, the sequence $\{J_n(x, y)\}$ must converge to a limit $J(x, y)$.

The stationarity of $J(x, y)$, $f(x, z)$, and $A(x, z)$ and the concavity of $J(x, y)$ and $f(x, z)$ imply that:

$$y_n^1(x) \to y^1(x) \quad \text{as} \qquad n \to \infty$$

$$y_n^2(x) \to y^2(x) \quad \text{as} \qquad n \to \infty$$

which completes the proof of part (b).

$\square$

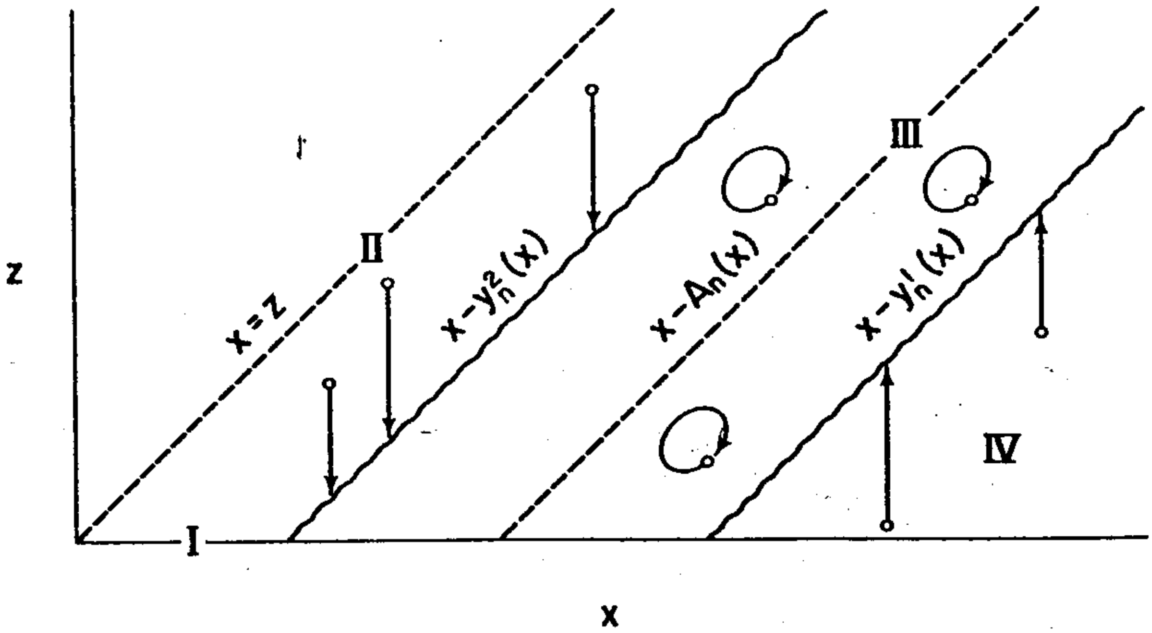

Figure 1.--This figure describes an optimal harvest for period n for a concave return function G(x - y). See text for more details.